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Uniqueness of the Critical Point of the Conformal Radius: "Method of Déjà vu"

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Abstract—New conditions are constructed for the critical point of the conformal radius (hyperbolic derivative) to be unique where the mapping function is holomorphic and locally univalent in the unit disk. We use an approach based on the uniqueness research of the univalence conditions depending on the additional parameters. Such a research has been carried out for the univalence criteria due to Singhs, Szapiel and some other mathematicians.

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1. INTRODUCTION

Jack's lemma in its original statement [1] asserts the existence of the real number k with the properties $k \geq 1$ and

$$\zeta_1 w'(\zeta_1) = k w(\zeta_1), \tag{1}$$

where $w(\zeta)$ is the holomorphic function in the disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with $w(0) = 0$, and the modulus of this function, $|w(\zeta)|$, attains its maximum value on the circle $|\zeta| = r(< 1)$ at a point $\zeta = \zeta_1$. On the base of this assertion it has been established in the article [2] that the condition

$$|w(\zeta)|^{1-\gamma} |\zeta w'(\zeta)|^{\gamma} < 1, \quad \zeta \in \mathbb{D}, \tag{2}$$

with some $\gamma \geq 0$ for the just mentioned $w(\zeta)$ implies the inequality $|w(\zeta)| < 1$, $\zeta \in \mathbb{D}$.

When

$$w(\zeta) = f'(\zeta) - 1, \tag{3}$$

the condition (2) gives the first of the theorems in the paper [2]. Namely, we have the following

Theorem S. *If a holomorphic function $f(\zeta)$ in $\mathbb{D}$ with the normalization*

$$f(0) = f'(0) - 1 = 0 \tag{4}$$

satisfies the condition

$$|f'(\zeta) - 1|^{1-\gamma} |\zeta f''(\zeta)|^{\gamma} < 1, \quad \zeta \in \mathbb{D}, \tag{5}$$

for some $\gamma \geq 0$, then $f(\zeta)$ is close-to-convex and bounded in $\mathbb{D}$.

We are interested in the uniqueness criteria for the critical point of the hyperbolic derivative

$$h_f(\zeta) = (1 - |\zeta|^2)|f'(\zeta)| \tag{6}$$

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